DIVERSIFICATION, REBALANCING, AND THE GEOMETRIC MEAN FRONTIER

William J. Bernstein

wbern@mail.coos.or.us

and

David Wilkinson

DaveWilk@worldnet.att.net

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Abstract

The effective (geometric mean) return of a periodically rebalanced portfolio always exceeds the weighted sum of the component geometric means. Some approximate formulae for estimating this effective return are derived and tested. One special case of these formulae is shown to be particularly simple, and is used to provide easily computed estimates of the benefits of diversification and rebalancing. The results are also used to show how classical Mean-Variance Optimization may be modified to generate the Geometric Mean Frontier, the analog of the efficient frontier when the geometric mean is used as the measure of portfolio return.

1 Introduction

The calculation of the true long term, or effective, return is an often ignored part of the portfolio optimization process. The case of U.S. common stocks and long term corporate bonds in Table 1 provides a well known example. Here the arithmetic mean return is simply the average of the 69 yearly returns, while the geometric mean return is the effective, or annualized, return over the entire period.

Let us consider an annually rebalanced portfolio consisting of equal parts of stocks and corporate bonds. Using a simple 50/50 average of the individual returns one obtains an anticipated portfolio return of 9.00 percent using the arithmetic mean returns and 7.85 percent using the geometric mean returns. In fact, neither is correct: A 50/50 portfolio, rebalanced annually, has an annualized return of 8.34 percent. To complicate matters further, if one had purchased the 50/50 mix on January 1, 1926 and not rebalanced, then by December 31, 1994 an almost 100 percent stock portfolio would have resulted, with an annualized return of 9.17 percent.

MacBeth (1995) recognized that the uncritical use of a weighted geometric mean results in an anticipated portfolio return which is too low, and suggested instead a formulation using the arithmetic mean corrected for variance. For the above example, this gives a return of 8.30 percent, very close to the actual rebalanced return of 8.34 percent.

The question of whether or not rebalancing benefits portfolio return is more complex. Perold and Sharpe (1995) examined the problem from the perspective of historical stock/bill returns and concluded:

> In general, a constant-mix (rebalanced) approach will underperform a comparable buy-and-hold (unrebalanced) strategy when there are no reversals. This will be the case in strong bull or bear markets, when reversals are small and relatively infrequent, because more of the marginal purchase and sell decisions will turn out to have been poorly timed.

We feel that this can only be part of the answer, and that another important criterion is whether or not the individual assets have similar long term return. Common experience demonstrates that rebalancing often yields significant excess returns when the return differences are small. Contrariwise, rebalancing penalizes the investor when asset return differences are large.

In this paper, we investigate the questions of rebalancing and long term portfolio return in a quantitative manner. We begin by discussing Mean-Variance Optimization (Markowitz (1952, 1991)) in the context of multi-period portfolio optimization. We then derive two families of approximate Portfolio Return Formulae, the first of which uses the individual arithmetic means as input, and the second the geometric means. As a special case of the latter, we obtain a simple approximate formula for the Diversification Bonus, the amount by which the geometric mean return of a rebalanced portfolio exceeds the weighted sum of the individual geometric means. This result is used to obtain a corresponding formula for the Rebalancing Bonus, the amount by which the return of the rebalanced portfolio exceeds that of the corresponding unrebalanced one. Lastly, we show how Mean-Variance Optimization may be modified to obtain the Geometric Mean Frontier (GMF), the analog of the efficient frontier when the geometric mean is used as the measure of portfolio return.

Maximization of the geometric mean return has been discussed extensively in the literature: Latane (1959), Hakansson (1971), Elton and Gruber (1974a, 1974b), Fernholz and Shay (1982). However most of this work was concerned with exact results, the question of whether maximizing the geometric mean can be justified on the basis of utility theory, or with the case of continuous time rebalancing. We take a more pragmatic and general approach: (a) we will be satisfied with approximate formulae for the portfolio geometric mean, (b) we consider the geometric mean in the context of actual historical data with a finite, but arbitrary, rebalancing interval, and (c) we focus on the entire Geometric Mean Frontier, as opposed to just the single portfolio which maximizes the geometric mean.

2 MVO – Past and Future

Mean-Variance Optimization (MVO) is designed to produce return/variance efficient portfolios. The standard Markowitz (1952, 1991) analysis is applicable to a single period, and the inputs are the expected returns R_i and covariance matrix V_{ij} for the individual assets over this period. The latter are related to the standard deviations s_i and correlation matrix ρ_{ij} by

$$V_{ij} = s_i s_j \rho_{ij} \quad , \tag{1}$$

where $\rho_{ij} = 1$ for $i = j$.

For a portfolio with fraction X_i assigned to asset i, with $\sum_i X_i = 1$, the expected return R and its variance V are given by

$$R = \sum_i X_i R_i \quad , \tag{2}$$

and

$$V = \sum_{ij} X_i X_j V_{ij} \quad . \tag{3}$$

The risk of the portfolio is taken to be the standard deviation $s = \sqrt{V}$.

One way to supply the inputs R_i and V_{ij} is to use historical returns, and to assume that the upcoming period will resemble one of the previous N periods, each with equal probability $1/N$. In this case, the expected return R_i becomes the arithmetic mean return of asset i over the N periods

$$R_i = \frac{1}{N} \sum_{k=1}^{N} r_i^{(k)} \quad , \tag{4}$$

where $r_i^{(k)}$ is the return of asset i in period k. For the portfolio, the return R in Eq. (2) becomes the arithmetic mean of the returns of a portfolio which is *rebalanced* to the mix specified by the X_i at the beginning of each period

$$R = \frac{1}{N} \sum_{k=1}^{N} r^{(k)} \quad , \tag{5}$$

where $r^{(k)} = \sum_i X_i r_i^{(k)}$ is the return of the rebalanced portfolio in period k. Note that while the period represented by the returns, e.g. annual or quarterly, is arbitrary, the rebalancing interval must always be the same as the measurement interval.

When viewed in this multi-period context, the Markowitz analysis is unsatisfactory because the long term return of an asset with returns $r^{(k)}$ in the different periods is given not by the arithmetic mean, Eq. (5), but rather by the geometric mean

$$G = \left[\prod_{k=1}^{N} (1 + r^{(k)}) \right]^{1/N} - 1 \quad . \tag{6}$$

Since the arithmetic mean of any return series is always greater than the geometric mean, the return predicted by the Markowitz analysis is always greater than the true long term return that would have been obtained by using the actual rebalanced allocation. For this reason the geometric mean returns of the individual assets G_i given as

$$G_i = \left[\prod_{k=1}^{N} (1 + r_i^{(k)}) \right]^{1/N} - 1 \tag{7}$$

are often used as inputs to the Markowitz analysis in preference to the arithmetic means R_i. However, as pointed out by MacBeth (1995), this is not correct either. As we shall see explicitly in the next section, this prescription always underestimates the true return of the rebalanced portfolio.

The discussion of the remainder of this paper will mostly be in terms of historical data, and the questions we will address are whether, given only partial information, it is possible to (a) estimate the true long term geometric mean return of a given rebalanced portfolio, (b) compare this return with that of the corresponding unrebalanced portfolio, and (c) estimate the composition of the rebalanced portfolio which maximizes the geometric mean return for a given level of risk. The partial information we consider is the MVO data alone, which for the case of historical data we will generalize to mean *either* the arithmetic mean return R_i *or* geometric mean return G_i for each asset, together with the covariance matrix V_{ij}. As in the classical situation, we identify the risk with the standard deviation, which for the case of historical data is a consequence of the fluctuation of the individual single period returns about their (arithmetic) mean value. The general conclusion is that, to a good approximation, all three goals above may be accomplished, and the reason we use the historical perspective is that it enables us to validate these conclusions. However, the purpose of portfolio theory is, of course, to provide a basis for decision-making for the future. In the case of the standard single period analysis, the input data are necessarily statistical in nature, and the covariance matrix represents the uncertainty in the returns for the single upcoming period. However, for multi-period forcasting, two entirely different viewpoints are possible.

In the first viewpoint, the investor seeks the best course of action over a chosen number of upcoming periods, based on the assumption that the hypothesized MVO data are actually realized. In this case the problem is conceptually no different from that with partial historical data. This viewpoint has no meaning in the case of the usual single period analysis, because if the return of each asset is assumed known, then the covariance matrix is zero, and the optimum strategy is just to select the asset with the highest return. However, in the multi-period case, the problem is not so simple. Suppose, as will usually be the case, that the input returns are chosen to be the geometric mean returns G_i. Firstly, the input covariance matrix represents not the uncertainty in these values, but rather the fluctuations of, and correlations between, the unspecified individual period returns which go to make up these values. Thus there is no contradiction between having both specified values for the returns and a non-zero correlation matrix. Secondly, the computational problem itself is non-trivial. Generally, as we shall see explicitly in Section 6, the rebalanced strategy with the highest geometric mean return is not to invest 100 percent in the asset with the highest geometric mean return; often the highest return strategy is to invest in a diversified portfolio containing several of the assets.

The second viewpoint is statistical in nature, and superficially more similar to the conventional single period one. It is, however, necessary to make some additional hypothesis about the time correlation of the returns. The most natural approach, which is in accord with the simplest form of the random walk theory of asset price movement, is to asssume that the distribution of returns is stationary, i.e. the same in each period, with each period being independent of the others. In this case the input return and covariance matrix represent the expected value and uncertainty of this unique single period distribution. The only new feature is that in the multi-period application we allow the possibilty of specifying the geometric means G_i of this distribution instead of the arithmetic means R_i. In this viewpoint, the significance of the geometric mean G (both for the individual assets, and for any rebalanced portfolio of them) is that, as the number of periods becomes large, it becomes increasingly probable that the actual long term return will lie close to G. This property is quite general, and is analogous to the observation that if a fair coin is thrown a large number of times, then it becomes increasingly likely that the fraction of heads will lie close to one half.

Which of these two viewpoints to adopt is perhaps a matter of personal taste. The actual mathematical development is the same in either case. We tend to prefer the first viewpoint, because it does not have to make any independent assumption about the time correlation of events. For example, if the proposed rebalancing period is annual, then the covariance matrix should correspond to annual returns; if it is quarterly, then the covariance matrix should correspond to quarterly returns. There is no need for any relationship to exist between these two covariance matrices. In the second viewpoint, however, consistency requires that the two covariance matrices be related by a factor of four, a relationship which is not necessarily satisfied in reality. For the purposes of this paper, it will certainly be simpler to think in terms of the first viewpoint, because there the future is treated in exactly the same way as the past.

3 Portfolio Return Formulae

In this section we derive and discuss a variety of approximate formulae for the geometric mean return of a balanced portfolio. These formulae express the portfolio geometric mean in terms of the arithmetic or geometric mean of the individual assets, together with their covariance matrix. In order to validate these formulae we will use an actual 9-asset global data set of annual index returns over the years 1970-1996. The arithmetic mean return, geometric mean return, and standard deviation of each asset are listed in Table 2; the correlation matrix was also computed but is not shown. Note that the asset with the highest arithmetic mean return (gold) is not the one which has the highest geometric mean return. In fact, both Japan and U.S. small stocks have higher long term return than gold over this period. Note also that, since these are annual returns, the rebalancing period we consider in the examples is also annual.

We first consider the weighted arithmetic mean and weighted geometric mean as possible portfolio return formulae. The results for the arithmetic mean are shown graphically in the top left plot of Fig. 1. As expected, we see that the weighted arithmetic mean overestimates the actual portfolio return in all cases. The corresponding results for the weighted geometric mean are shown in in the top right plot of Fig. 1, where we see that, in contrast, the geometric mean of the portfolio is always underestimated. Neither the weighted arithmetic mean or weighted geometric mean are adequate models for the portfolio geometric mean.

Improved geometric mean formulae may be based on the well known approximate relationship between the geometric mean G and arithmetic mean R of any set of returns

$$G \approx R - \frac{V}{2(1+R)} \quad , \tag{8}$$

where V is the variance of the returns (see, for example, Markowitz (1991)). This formula strictly holds in the limit where the variance is small, but it is surprisingly accurate even outside this range. If, in addition, the arithmetic mean return is small compared to unity then this reduces to

$$G \approx R - \frac{V}{2} \quad . \tag{9}$$

It will be useful in the following to regard both these formulae as special cases of

$$G \approx R - \frac{\alpha V}{2(1+\beta R)} \quad , \tag{10}$$

which we will refer to as the (α, β) formula. In this paper we will only consider the cases $(1, 0)$ and $(1, 1)$, but the general (α, β) case is no more difficult to analyze than $(1, 1)$, so most of the results will apply to this general case.

A useful feature of the (α, β) formula is that it may be inverted to give

$$R = \frac{2G + \alpha V}{1 - \beta G + \sqrt{(1+\beta G)^2 + 2\alpha\beta V}} \quad . \tag{11}$$

When $\beta = 0$ this simplifies to

$$R = G + \frac{\alpha V}{2} \quad . \tag{12}$$

We first explore the strategy of simply applying the (α, β) formula to the portfolio. We will call this method $\mathrm{A}(\alpha, \beta)$, where the 'A' signifies that the Arithmetic means R_i of the individual assets are used to construct the arithmetic mean $R = \sum_i X_i R_i$ of the portfolio. The portfolio variance is obtained from

$$V = \sum_{ij} X_i X_j V_{ij} \quad . \tag{13}$$

The results for methods A(1,0) and A(1,1) are shown in the left column of Fig. 1. It is seen that in both cases there is an improvement over the simple weighted arithmetic or geometric means. The A(1,1) formula in particular does extremely well for the two-asset and randomly selected portfolios, and only fails for the high return single asset portfolios. Even here, however, the results for both A(1,0) and A(1,1) are much better than those obtained using simple weighted arithmetic or geometric means. Note that the weighted arithmetic mean may be considered as the A(0,0) case of $\mathrm{A}(\alpha, \beta)$, so that the three plots on the left side of Fig. 1 may be viewed as three cases of the same formula, with the accuracy impoving from top to bottom.

Two drawbacks of the $\mathrm{A}(\alpha, \beta)$ method are (a) the formula uses the arithmetic means of the individual assets rather than their geometric means, and (b) the formula does not give the correct result in the simplest case where the portfolio consists of only a single asset. Both these defects may be remedied by applying the (α, β) formula not only to the portfolio but also to the individual assets. To do this we first use the inverse relationship on the individual assets to obtain "pseudo-arithmetic means"

$$R_i^* = \frac{2G_i + \alpha V_{ii}}{1 - \beta G_i + \sqrt{(1 + \beta G_i)^2 + 2\alpha\beta V_{ii}}} \quad , \tag{14}$$

and then compute the portfolio arithmetic mean using $R = \sum_i X_i R_i^*$ before using the (α, β) formula for the portfolio. We call this method $\mathrm{G}(\alpha, \beta)$, where the 'G' signifies that the geometric means of the individual assets are used as inputs. The results for G(1,0) and G(1,1) are shown in the right column of

Fig. 1. It is seen that while the results for the two-asset and random portfolios are not as good as for the corresponding A(1,0) and A(1,1) cases, the results for the single asset portfolios are now exact. The weighted geometric mean may be considered as the G(0,0) case of G(α, β), so that the three plots on the right side of Fig. 1 may be viewed as three cases of the same formula, with the accuracy improving from top to bottom.

In this section we have derived two families A(α, β) and G(α, β) of portfolio return formulae, the first of which requires the individual arithmetic means as inputs and the second their geometric means. While we have focused on the two cases (1,1) and (1,0), which have some theoretical foundation, it is possible that, empirically, values other than these might prove to be more accurate.

4 The Diversification Bonus

The general G(α, β) method succeeds in expressing the geometric mean return of the portfolio in terms of the geometric mean returns of the individual assets. Though simple to evaluate numerically, the expession is rather cumbersome and not very intuitive. However, when $\beta = 0$ the expression simplifies because the inverse relation in Eq. (11) reduces to the simpler Eq. (12). In the following, we will also set $\alpha = 1$, though the case of general α is no more difficult. In this case, $\mathrm{G}(1, 0)$, we find

$$G \approx \sum_i X_i G_i + \frac{1}{2}\left(\sum_i X_i V_{ii} - \sum_{i,j} X_i X_j V_{ij}\right) \quad . \tag{15}$$

The analogous result in the limit of continuous time rebalancing has been obtained previously by Fernholz and Shay (1982), who showed that it is exact when the prices follow geometric Brownian motion. A different form of the same result may be obtained by using $\sum_i X_i = 1$ in the first term inside the parentheses and symmetrizing to give

$$G \approx \sum_i X_i G_i + \sum_{i<j} X_i X_j \left(\frac{V_{ii}}{2} + \frac{V_{jj}}{2} - V_{ij}\right) \quad , \tag{16}$$

where the sum is over all pairs i and j with $i < j$.

This form of the G(1, 0) formula has been obtained previously using an empirical argument [1]. The interesting feature of this formula, not shared by the general G(α, β) with $\beta \neq 0$, is that it separates into two terms, the first of which depends only on the geometric mean returns, and the second only on the covariance matrix. Eq. (16) demonstrates that, for given values of the individual returns G_i, it is possible for the portfolio return G to be an increasing function of the volatility (standard deviation) of the individual assets. In particular, if all the off-diagonal elements of the correlation matrix ρ_{ij} are zero or negative, then the portfoloio return is an increasing function of each of the standard deviations s_i. The benefits of a rebalancing strategy become much greater for assets which are volatile and poorly correlated.

Let us write Eq. (16) in the form

$$G - \sum_i X_i G_i \approx \sum_{i<j} X_i X_j \left(\frac{V_{ii}}{2} + \frac{V_{jj}}{2} - V_{ij} \right) \quad . \tag{17}$$

We will refer to the left side of Eq. (17), which is always positive, as the Diversification Bonus [2], because it represents the effective return in excess of that calculated from the simple combination of the individual geometric mean returns. The right hand side of Eq. (17) provides an approximation to this quantity which is expressed solely in terms of the covariance matrix of the assets. On the left side of Fig. 2 we compare the approximate formula, Eq. (17), with the true value of $G - \sum_i X_i G_i$. It is seen that some of the portfolios have a Diversification Bonus in excess of 200 basis points. Although Eq. (17) tends to overestimate the precise value of $G - \sum_i X_i G_i$, as we would expect from the plot for formula G(1,0) in Fig. 1, it does a good job of predicting which portfolios will have significant Diversification Bonus. On the right side of Fig. 2 we show the corresponding plot for the case G(1,1); while this formula does not lead to a simple form like Eq. (17), it gives a better approximation to the true Diversification Bonus, as again we would expect from Fig. 1.

5 The Rebalancing Bonus

The weighted geometric mean $\sum_i X_i G_i$ provides a naive estimate of effective portfolio return, but it does not correspond to the return obtainable by any particular investment strategy. A more meaningful comparison is that between the return G of the rebalanced portfolio and the return G' of the corresponding unrebalanced portfolio. The latter is given by

$$G' = \left[\sum_i X_i (1 + G_i)^N \right]^{1/N} - 1 \quad . \tag{18}$$

Subtracting Eq. (18) from Eq. (16) we obtain

$$\begin{aligned} G - G' \;\; &\approx \;\; \left[\sum_i X_i (1 + G_i) \right] - \left[\sum_i X_i (1 + G_i)^N \right]^{1/N} \\ &\qquad + \sum_{i<j} X_i X_j \left(\frac{V_{ii}}{2} + \frac{V_{jj}}{2} - V_{ij} \right) \quad . \end{aligned} \tag{19}$$

We will refer to the left side of Eq. (19) as the Rebalancing Bonus[3], because it represents the difference between the returns of the rebalanced and unrebalanvced portfolios. In the approximation on the right hand side, the term on the first line always gives a negative contribution, thus favoring the unrebalanced portfolio. This term is large when the differences between the long term returns G_i are large, and vanishes when all the returns G_i are the same. The term in the second line always gives a positive contribution, thus favoring the rebalanced portfolio. This term is large when the individual assets have large variances, and low or negative correlation.

For the data set considered in Section 3, it is found that the majority of portfolios perform better when rebalanced. This is illustrated in the left side of Fig. 3, in which we compare Eq. (19) with the true value of $G - G'$. It is seen that the great majority of the portfolios have positive values for this quantity. Examination of the data underlying Fig. 3 shows that the only portfolios giving a significant negative value are 50/50 2-asset portfolios in which one of the assets is Treasury Bills. This is in accordance with the above discussion – only in this case is the return difference sufficient to overcome the second term in Eq. (19). We also see that the approximate formula in Eq. (19)

always gives the correct sign for $G-G'$, although it tends to overestimate the precise benefit of rebalancing, as we would expect from the plot for formula G(1,0) in Fig. 1. On the right side of Fig. 3 we show the corresponding plot for the case G(1,1); while this formula does not lead to the simplifying separation of Eq. (19), it gives a better model value for $G - G'$, as again we would expect from Fig. 1.

The first line in Eq. (19) vanishes when all the returns G_i are equal, but there is no simple way to express Eq. (19) in terms of return differences alone. Over a long enough time period, however, the unrebalanced portfolio becomes dominated by the highest return asset, independent of the initial allocation, and we obtain

$$G' \approx G_{\max} \quad , \tag{20}$$

where $G_{\max} = \max_i G_i$ is the maximum return obtainable from any unrebalanced portfolio. In this case the Rebalancing Bonus becomes

$$G - G' \approx \sum_i X_i(G_i - G_{\max}) + \sum_{i<j} X_i X_j \left(\frac{V_{ii}}{2} + \frac{V_{jj}}{2} - V_{ij} \right) \quad . \tag{21}$$

In this case it is clear that if all the G_i are the same, then the rebalanced portfolio is always superior to the unrebalanced one.

6 The Geometric Mean Frontier

When used with historical data, the traditional Markowitz efficient frontier designates those portfolios with greater (arithmetic mean) return than any other with the same or lesser risk, and lesser risk than any other with the same or greater return. The Markowitz return R and risk s are defined by

$$R = \sum_i X_i R_i \quad , \tag{22}$$

and

$$s^2 = \sum_{ij} X_i X_j V_{ij} \quad . \tag{23}$$

In this section we show that it is possible to modify the MVO analysis to construct the analogous efficient frontier when the geometric mean is used instead of Eq. (22) as the measure of return, while keeping Eq. (23) as the measure of risk. More precisely, we show that this Geometric Mean Frontier (GMF) may be computed exactly when any of the approximate geometric mean formulae A(α, β) or G(α, β) of Section 3 is used, and that in each case the efficient frontier so obtained provides a good approximation to the efficient frontier of the true geometic mean, the latter being obtained using the optimization feature of a standard spreadsheet package[4].

In each of the methods A(α, β) or G(α, β), the portfolio geometric mean has the form

$$G = R - \frac{\alpha s^2}{2(1 + \beta R)} \quad , \tag{24}$$

where R and s are given in terms of the individual assets by expressions of the form of Eqs. (22) and (23). The only difference between the cases is that for A(α, β) the individual returns R_i are the true arithmetic mean returns, while for G(α, β) the R_i are pseudo-arithmetic means R_i^* given in Eq. (14). To simplify the notation we will omit the superscript in this latter case. Also, in this section we will use the symbol 'G' to denote the appropriate model geometric mean, so that Eq. (24) should be thought of as a definition, rather than as an approximate relation for the true geometric mean.

The key to the analysis is to consider an auxiliary classical Markowitz optimization which uses the *arithmetic* means as inputs. We emphasize that even in the case of G(α, β), which expresses the portfolio geometric mean in terms of the individual geometric means, the inputs to this auxiliary Markowitz problem are the individual pseudo-arithmetic means, not the geometric means.

We begin by generalizing an argument first given by Elton and Gruber (1974b) to show that any portfolio which is (G, s) efficient must also be Markowitz (R, s) efficient. For any (α, β), the geometric mean expression in Eq. (24) has the properties

$$\frac{\partial G}{\partial R} > 0 \quad , \tag{25}$$

$$\frac{\partial G}{\partial s} < 0 \quad . \tag{26}$$

Suppose that (R_1, s_1), with corresponding geometric mean G_1, is not Markowitz efficient. Then there exists a portfolio (R_2, s_2), with corresponding geometric mean G_2, with either $R_2 > R_1$ and $s_2 \leq s_1$, or $R_2 \geq R_1$ and $s_2 < s_1$, or both. In either case it follows from Eqs. (25) and (26) that $G_2 > G_1$ and $s_2 \leq s_1$. Thus no portfolio which is not Markowitz efficient can be geometric mean efficient; equivalently, every geometric mean efficient portfolio must be Markowitz efficient. Note that the above argument does not assume the existence of a continuous path of portfolios between portfolios 1 and 2.

The Markowitz efficient frontier extends between the minimum variance portfolio at $s = s_{\text{min}}$ and the maximum return portfolio at $s = s_{\text{max}}$. For each s in the range $s_{\text{min}} \leq s \leq s_{\text{max}}$ the frontier gives a corresponding return $R_f(s)$, and this function $R_f(s)$ has the properties

$$\frac{\mathrm{d}R_f}{\mathrm{d}s} > 0 \quad , \tag{27}$$

$$\frac{\mathrm{d}^2 R_f}{\mathrm{d}s^2} < 0 \quad . \tag{28}$$

Let us define $G_f(s)$ by evaluating Eq. (24) on the Markowitz frontier, i.e.

$$G_f(s) = R_f(s) - \frac{\alpha s^2}{2(1 + \beta R_f(s))} \quad . \tag{29}$$

From the above discussion, the (G, s) efficient frontier must be a sub-graph of the graph of $G_f(s)$. Thus the multi-dimensional space of candidate (G, s) efficient portfolios has been reduced to a one-dimensional one.

It can be shown using Eqs. (27) and (28) that, under very reasonable assumptions on the nature of the function $R_f(s)$, the function $G_f(s)$ can have only one maximum in the range $s_{\text{min}} \leq s \leq s_{\text{max}}$. In general there are three possibilities: (a) the maximum occurs at s_{min}; in this case the geometric mean frontier consists of a single portfolio, the minimum variance portfolio, (b) the maximum occurs at s_{max}; in this case the entire Markowitz frontier is geometric mean efficient, or (c) the maximum occurs at some intermediate point; in this case only that part of the Markowitz frontier to the left of this maximum is geometric mean efficient. This third possibility is the one which occurs for our chosen example.

The portfolio with the maximum geometric mean may be algorithmically determined by starting from the minimum variance portfolio at $s = s_{\min}$ and moving to the right along the Markowitz frontier until the corresponding $G = G_f(s)$ first decreases. The standard MVO output consists of a set of "corner portfolios" located on the Markowitz frontier, between each adjacent pair of which the frontier is obtained by linear combination. The task of testing for a maximum of $G_f(s)$ between any pair of corner portfolios is easily accomplished by using the composition, rather than the risk, as the independent variable. Since $G_f(s)$ has only one maximum, a maximum will be found in at most one corner interval, and that interval will always be one which is adjacent to the corner portfolio with the highest geometric mean return.

We examine the validity of these ideas using the same 9-asset data set considered in Section 3. In Fig. 4 we plot the Markowitz efficient frontier, and associated model and actual geometric means, for the four cases A(1,0), G(1,0), A(1,1) and G(1,1). The result of the exact spreadsheet optimization of the true geometric mean is also shown. The four cases are qualitatively very similar. In each, the maximum of the geometric mean (model or actual) occurs at an intermediate value of risk; the Markowitz frontier greatly overestimates the true reward of increasing risk. Note also that, while the model returns in the four cases show some differences, as we would expect from the discussion of Section 3, the true geometric mean evaluated on the frontier is for each case in excellent agreement with the result of the full optimization of the true geometric mean.

In Table 3 we present in tabular form three portfolios on the efficient frontier for each of the the four methods A(1,0), G(1,0), A(1,1) and G(1,1), together with the exact numerical result. Table 3a shows the portfolio which maximizes the model geometric mean. We see that the composition of the optimum portfolio is in all four model cases in reasonable agreement with the exact one, and that despite the fact that the corresponding model geometric means are somewhat different from each other, the true geometric means are close not only to each other, but also to the correct maximum of 16.85 percent. An even better result may be obtained by choosing the optimal portfolio to be that which maximizes (over the Markowitz frontier) the true geometric mean return, rather than the model geometric mean. The corresponding results are shown in Table 3b. In this case it is seen that the portfolio compositions are in closer agreement with the exact ones, and that, to the quoted precision, all four approximate methods obtain the correct maximum geometric mean of 16.85 percent. It is also possible to use the efficient frontier of the auxiliary Markowitz problem to determine the geometric mean efficient portfolio corresponding to a given value of risk s, or model geometric mean G, or true geometric mean. As an example, we show in Table 3c the results obtained by fixing s to be 10.0 percent. Again, we see that the portfolio compositions are in good agreement with the exact ones, and that the values for the true geometric mean reproduce the exact value of 13.54 percent to within at most one basis point.

The extremely close agreement when the true geometric mean return is used follows from a general result of perturbation theory, namely that when the objective function of an optimization is perturbed by a small amount, the change in the value of the unperturbed objective function in going to the new maximum is second order in the perturbation. The excellent agreement is thus due to the fact that all the $\mathrm{A}(\alpha, \beta)$ and $\mathrm{G}(\alpha, \beta)$ formulae are reasonable approximations to the true geometric mean.

The net result of this section is that, although the $\mathrm{A}(\alpha, \beta)$ and $\mathrm{G}(\alpha, \beta)$ formulae are not perfect models for the true geometric mean, in each case the efficient frontier of the auxiliary Markowitz problem does contain portfolios whose true geometric mean return is extremely close to the real maximum, either in an absolute sense, or when the risk is specified to take a given value. Thus, finding the optimal geometric mean portfolio becomes a tractable numerical task, even for a large number of assets, because the space of candidate portfolios becomes one-dimensional.

7 Discussion

This paper contains four significant results, each of which can be easily used by investors who are able to express their data in the form commonly used by a standard Mean-Variance Optimization package – namely the arithmetic mean return R_i or geometric mean return G_i of each asset, and the covariance matrix V_{ij} between the assets. These results are:

1. Two families $A(\alpha, \beta)$ and $G(\alpha, \beta)$ of approximate Portfolio Return Formulae for estimating the geometric mean return of a rebalanced portfolio. The family $A(\alpha, \beta)$ requires the arithmetic means of the individual assets, and $G(\alpha, \beta)$ the geometric means.

2. For the case G(1,0), a particularly simple approximate formula, Eq. (17), for the Diversification Bonus, the amount by which the portfolio geometric mean exceeds the weighted sum of the individual geometric means.

3. A corresponding approximate formula, Eq. (19), for the Rebalancing Bonus, the difference between the returns of the rebalanced and unrebalanced portfolios.

4. An extension of classical Mean-Variance Optimization which allows construction of a good approximation to the entire Geometric Mean Frontier (GMF), and in particular that portfolio which maximizes the geometric mean return.

The validity of these results was demonstrated using historical data. The chosen data set, containing as it does some extremely volatile assets, is a stringent test of the theory; much more accurate results would be obtained by applying the methodology to a less volatile group of assets, such as different classes of U.S. stocks and bonds. However we believe that the major application of these ideas is precisely in generating diversified portfolios of very volatile assets, because there lies the greatest potential benefit of diversification and rebalancing, and of optimizing specifically for long term return.

The formalism presented here is not limited to annual returns, but may be applied for any rebalancing period. For example, if the rebalancing period is quarterly, then the arithmetic and geometric mean returns should be quarterly returns, and the covariance matrix should be constructed from the historical quarterly returns. If desired, the resulting portfolio return G may be converted to annual return at the end of the calculation. The methodology may thus be used to investigate the effect of varying the rebalancing period.

While our focus has been on the analysis of historical data, the goal of portfolio theory is of course to generate optimal strategies for the future. In the historical case, the simplest and most accurate of the MVO methods presented here is to use the arithmetic means as inputs to the optimizer, and use the A(1,1) formula or, even better, the actual returns, to evaluate the portfolio geometric mean along the frontier. This will always generate an excellent approximation to the true Geometric Mean Frontier. There is no need to consider the $G(\alpha, \beta)$ formulae and pseudo-arithmetic means. However, when considering long term future strategies the situation is very different (for the interpretation of the formalism in this situation, recall the discussion at the end of Section 2). Here it is unlikely that investors will simply adopt the historical geometric mean, arithmetic mean and covariance matrix. At the least, they will provide their own estimate of return, and this return will be the effective long term, i.e. geometric mean, return. In this case the $G(\alpha, \beta)$ family of Portfolio Return Formulae, and the associated use of pseudo-arithmetic means as MVO inputs, become indispensible. The G(1,1) formula will give the best results when used in the MVO analysis, while the simpler G(1,0) formula can provide good intuitive understanding of the performance of different portfolio allocations. We emphasize once again that simply using the individual geometric means as MVO inputs always underestimates the true long term return, even when the input data for the individual assets prove to be correct.

Notes

1. Bernstein, William J., "The Rebalancing Bonus: Theory and Practice", Efficient Frontier (September 1996), http://www.coos.or.us/~wbern/ef.
2. Previously (see Note 1) the left side of Eq. (17) was termed the Rebalancing Bonus; here we use this expression for the left side of Eq. (19).
3. See Note 2.
4. Quattro Pro, Version 6.0.

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	Arithmetic mean	Geometric mean	Standard deviation
STOCKS	0.1216	0.1019	0.2020
BONDS	0.0583	0.0551	0.0850

Table 1: Arithmetic mean, geometric mean and standard deviation for U.S. Common Stocks and Long Term Corporate Bonds for 1926-94.

	Arithmetic mean	Geometric mean	Standard deviation
S&P 500	0.1346	0.1227	0.1585
SMALL US (9-10)	0.1659	0.1415	0.2293
EAFE-E	0.1486	0.1305	0.2095
EAFE-PXJ	0.1641	0.1226	0.3084
JAPAN	0.1895	0.1454	0.3368
GOLD	0.2014	0.1370	0.4299
20 Y TREAS	0.0989	0.0927	0.1189
5 Y TREAS	0.0950	0.0928	0.0686
T BILL	0.0692	0.0688	0.0267

Table 2: Annual arithmetic mean return, geometric mean return, and standard deviation for nine asset classes over the period 1970-1996. *S&P 500, U.S. 9-10, and Treasury security retrurns from SBBI, Ibbotson and Associates. MSCI-EAFE-E, MSCI-EAFE-PXJ and MSCI Japan from Morgan Stanley. Gold from Morningstar Inc, and Van Eck Group.*

	A(1,0)	G(1,0)	A(1,1)	G(1,1)	Exact
SMALL US (9-10)	0.3530	0.2190	0.3049	0.2204	0.2767
JAPAN	0.3385	0.3778	0.3633	0.3866	0.3533
GOLD	0.3085	0.4032	0.3318	0.3931	0.3699
Markowitz return	0.1848	0.2056	0.1863	0.1946	0.1874
Standard deviation	0.1969	0.2242	0.2047	0.2228	0.2120
Model geometric mean	0.1655	0.1805	0.1686	0.1738	0.1685
True geometric mean	0.1681	0.1683	0.1684	0.1684	0.1685

(a) Maximum model geometric mean

	A(1,0)	G(1,0)	A(1,1)	G(1,1)	Exact
SMALL US (9-10)	0.2649	0.2769	0.2649	0.2732	0.2767
JAPAN	0.3839	0.3526	0.3839	0.3623	0.3533
GOLD	0.3512	0.3705	0.3512	0.3646	0.3699
Markowitz return	0.1874	0.2027	0.1874	0.1925	0.1874
Standard deviation	0.2120	0.2120	0.2120	0.2119	0.2120
Model geometric mean	0.1650	0.1803	0.1685	0.1736	0.1685
True geometric mean	0.1685	0.1685	0.1685	0.1685	0.1685

(b) Maximum true geometric mean

	A(1,0)	G(1,0)	A(1,1)	G(1,1)	Exact
SMALL US (9-10)	0.2407	0.2218	0.2407	0.2270	0.2396
EAFE-E	0.0172	0.0096	0.0172	0.0186	0.0166
JAPAN	0.1402	0.1445	0.1402	0.1414	0.1400
GOLD	0.1290	0.1403	0.1290	0.1359	0.1303
5 Y TREAS	0.4729	0.4839	0.4729	0.4772	0.4736
Markowitz return	0.1399	0.1461	0.1399	0.1416	0.1399
Standard deviation	0.1000	0.1000	0.1000	0.1000	0.1000
Model geometric mean	0.1349	0.1411	0.1355	0.1372	0.1354
True geometric mean	0.1354	0.1353	0.1354	0.1354	0.1354

(c) Portfolio with standard deviation $s = 0.10$

Table 3: Three portfolios on the Markowitz frontier for each of the four cases A(1,0), G(1,0), A(1,1) and G(1,1). The final column shows the corresponding exact result.

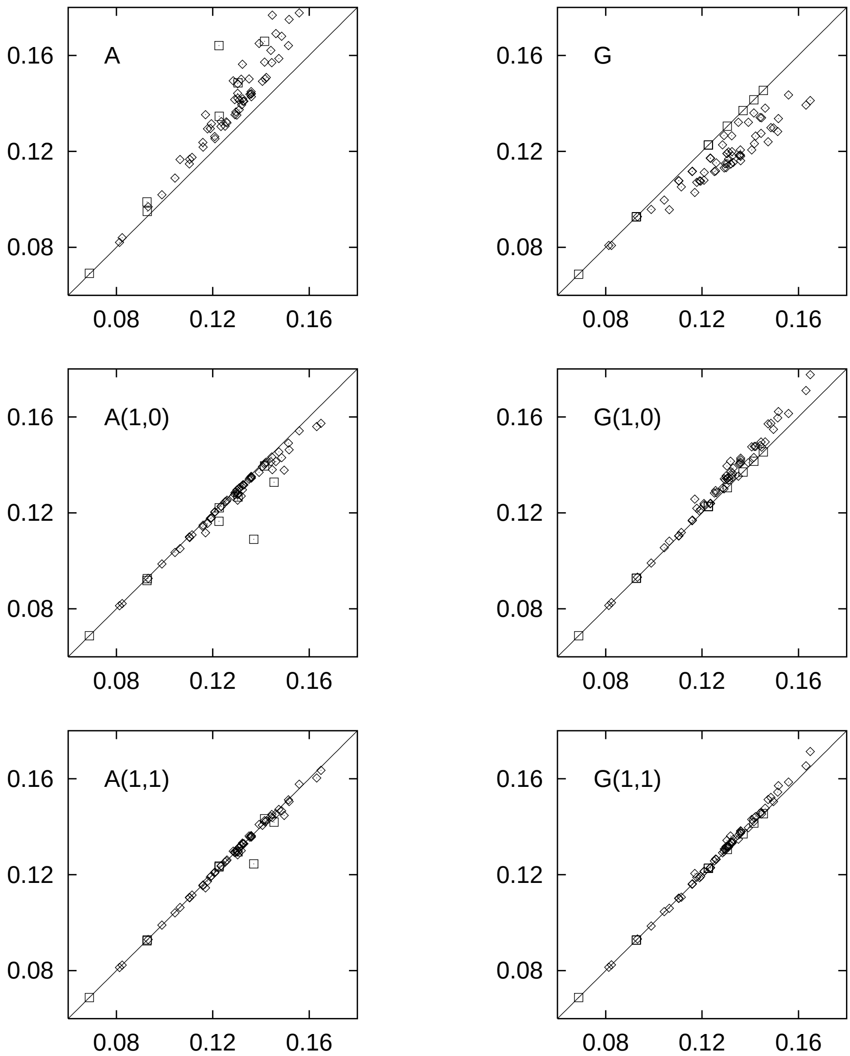

Figure 1: Cross-plot of model (vertical axis) against actual (horizontal axis) geometric mean return, for weighted arithmetic mean (A), weighted geometric mean (G), and formulae A(1,0), G(1,0), A(1,1) and G(1,1). Each plot shows the 9 single-asset portfolios (squares), the 36 two-asset portfolios with 50-50 mix, and 25 randomly generated portfolios.

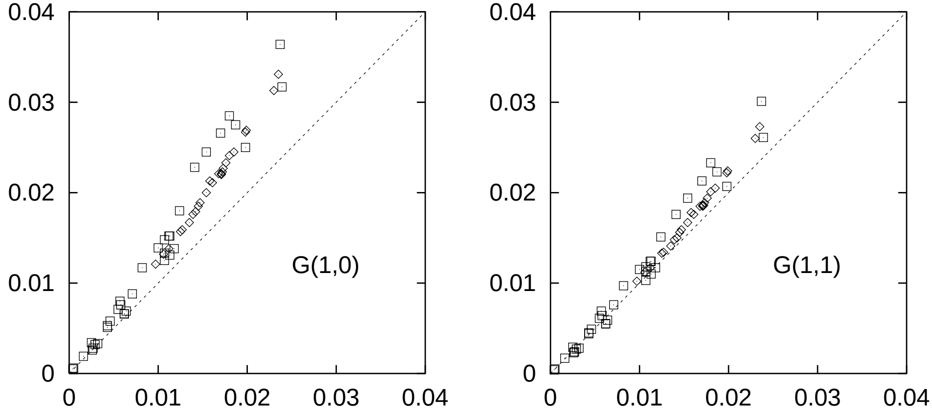

Figure 2: Cross-plot of model (vertical axis) against actual (horizontal axis) Diversification Bonus, $G - \sum_i X_i G_i$, using formulae G(1,0) and G(1,1) for the model return. Each plot shows the 36 two-asset portfolios with 50-50 mix (squares), and 25 randomly generated portfolios (diamonds).

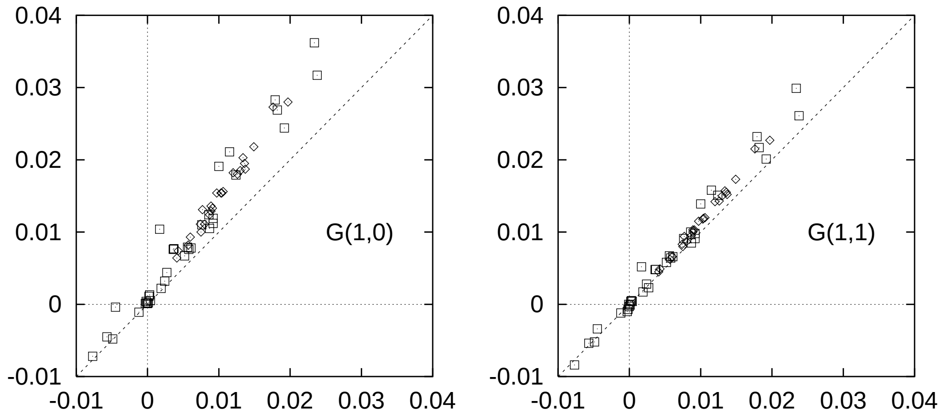

Figure 3: Cross-plot of model (vertical axis) against actual (horizontal axis) Rebalancing Bonus, $G - G'$, using formulae G(1,0) and G(1,1) for the model rebalanced return. Each plot shows the 36 two-asset portfolios with 50-50 mix (squares), and 25 randomly generated portfolios (diamonds).

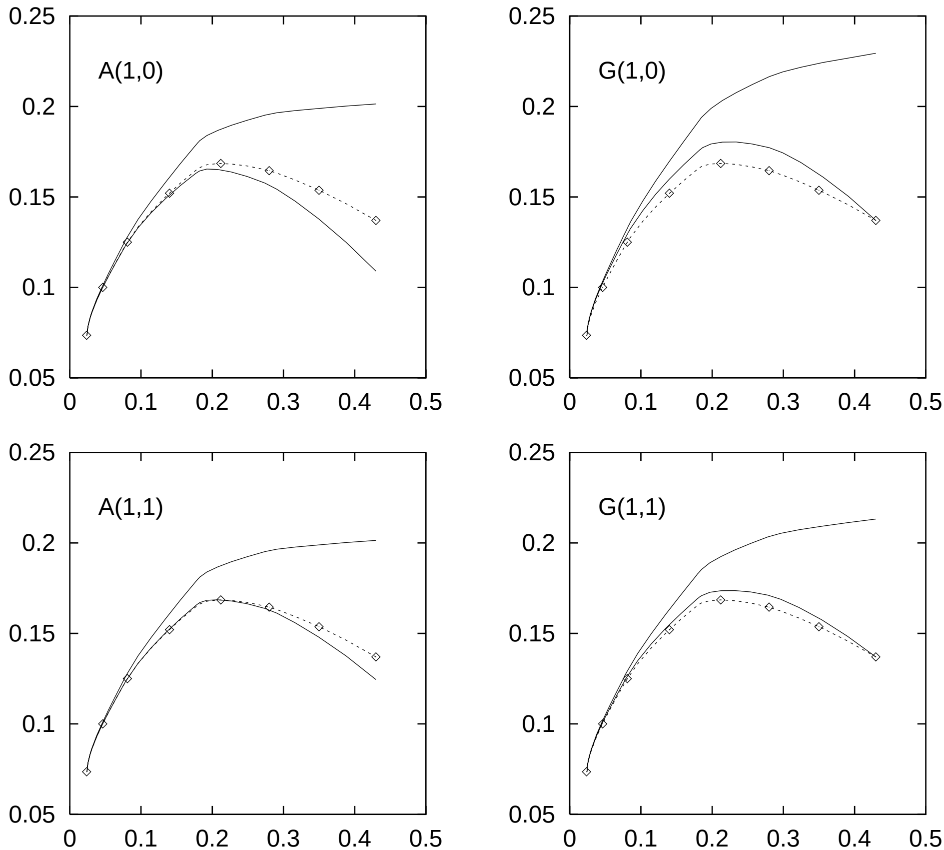

Figure 4: Risk-return plot for each of the four cases A(1,0), G(1,0), A(1,1) and G(1,1). In each plot the upper solid curve is the Markowitz efficient frontier, the lower solid curve the corresponding value of the model geometric mean, and the dotted curve the corresponding value of the true geometric mean. The open diamonds denote the exact optimized return for each value of risk.